In order to exploit symmetry analysis fully, one must use "common sense" in applying it to particular structures. For example, suppose the waveguide under consideration has a particular symmetry type, but its cross section is such that it "almost" has a higher symmetry type. This waveguide may well have mode classes which are nearly degenerate, and one would be advised to study the implications of both symmetry types to predict the modal characteristics the structure would exhibit. Actually, a deeper exploration of symmetry analysis can indicate how the degeneracies of modes are split when the symmetry is "lowered;" this would require some knowledge of group representation theory and is not considered here.

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Symmetry-Induced Modal Characteristics of Uniform Waveguides—II: Theory

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Abstract—**The application of symmetry analysis to uniform waveguides is discussed. Symmetry analysis provides exact information concerning mode classification, mode degeneracy, modal electromagnetic-field symmetries, and the minimum waveguide sectors which completely determine the modes in each mode class. This paper provides a summary of the development that leads to the results concerning symmetry-induced modal characteristics of uniform waveguides discussed in the previous paper. Some of the concepts of group theory are introduced, including the irreducible representations of symmetry groups. The use of the irreducible representations to determine the mode classes and their degeneracies is described. The projection operators belonging to the irreducible representations are introduced and their application to determining the azimuthal symmetry of the modal fields is explained. The minimum waveguide sectors for the mode classes are obtained from the azimuthal symmetry of the modal fields.**

I. INTRODUCTION

THE PURPOSE of this paper is to provide a summary of the development that leads to the results concerning the symmetry-induced modal characteristics of uniform waveguides discussed in the previous paper. These results are based on group theory and, in particular, on the theory of group representations. There have been many applications of group theory to various branches of physics and chemistry, and the literature describing these applications is copious. However, there have been few applications of group theory to the field of microwaves. One exception is symmetrical waveguide junctions which have been investigated by Montgomery *et al.* [1], Kerns [2], and Auld [3]. A few papers have been published which explored the consequences of symmetry in periodic waveguides. Two recent publications are [4] and [5]; the second paper employs group-theoretic methods. There has been little attention given, however, to exploiting the role symmetry plays in determining the modal characteristics of uniform waveguides.

A coherent exposition of the development of the complete theory required for the symmetry analysis of uniform waveguides starting from the basic concepts of group theory is not feasible in the few pages appropriate to a journal paper, and this is not attempted here. Instead, the relevant results from group theory will be cited, and a brief indication given how these lead to the results presented for uniform waveguides in the previous paper (hereafter referred to as [I]). This paper is not intended to enable a reader unfamiliar with group theory to attain a working knowledge of it as a technique for application to microwave analysis. However, it is hoped that these papers may provide a glimpse of the power of this technique and motivate some readers to explore it. Three of the many excellent books on the application of group theory to various branches of physics and chemistry are [6]–[8]. To provide the maximum assistance to any interested

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reader of this abbreviated account, specific page references will be made to the book by Tinkham [8] for the group theoretical results needed in this exposition.

II. UNIFORM-WAVEGUIDE EQUATIONS

In these papers attention is restricted to uniform waveguides which may be transversely inhomogeneous, but whose media are isotropic and piecewise homogeneous.[1] For a uniform waveguide of infinite length, and assuming an exp $(j\omega t)$ time dependence, the possible electromagnetic fields can be classified into a set of modes, each of which varies as exp $(-\gamma z)$, where the propagation constant γ is characteristic of the mode and a function of ω. For waveguides with a closed boundary, the mode spectrum is discrete, and there are an infinite set of discrete values for each ω. For open boundary waveguides, the mode spectrum consists of a finite set of discrete modes plus a continuous spectrum.

The transverse components of the electromagnetic fields of any mode can be expressed in terms of the longitudinal components [9]. In the ith medium of an inhomogeneous waveguide the transverse components can be written as

$$E_{Ti} = \frac{-1}{\gamma^2 + k_i^2} \{\gamma \nabla_T E_{zi} - jk_i Z_i (a_z \times \nabla_T H_{zi})\}$$

$$H_{Ti} = \frac{-1}{\gamma^2 + k_i^2} \{\gamma \nabla_T H_{zi} + j(k_i/Z_i)(a_z \times \nabla_T E_{zi})\}.$$

Here, $k_i = \omega(\mu_i \epsilon_i)^{1/2}$ and $Z_i = (\mu_i/\epsilon_i)^{1/2}$ are parameters characteristic of the ith medium, ∇_T is the transverse ∇ operator, and a_z is a unit vector in the z direction.

The partial differential equations for the longitudinal components of the electric and magnetic fields in the ith medium are

$$[\nabla_T^2 + k_i^2]E_{zi} = -\gamma^2 E_{zi}$$

$$[\nabla_T^2 + k_i^2]H_{zi} = -\gamma^2 H_{zi}.$$

The boundary conditions at the interfaces between the different media must also be considered. These boundary conditions are based on the continuity of the tangential components of the electric and magnetic fields at the interfaces. The boundary conditions at the surrounding waveguide wall (if any is present) must also be included. If the waveguide has an open boundary, then the modal fields must fall off at least as fast as $1/r^{1/2}$, for large values of the radius r.

The set of partial differential equations for E_z and H_z for all regions of the waveguide, together with the set of boundary conditions, form an eigenvalue problem. For a given value of the frequency ω, the set of allowed values of γ are the eigenvalues, and the corresponding pairs of E_z, H_z are the eigenfunctions.

For the purposes of symmetry analysis, it is not necessary to find explicit solutions to the eigenvalue problem summarized here. Several of the modal characteristics can be deduced from the symmetry of the waveguide cross section alone. The modes of inhomogeneous waveguides are, in general, hybrid modes with longitudinal components of both the electric and magnetic fields. Homogeneous waveguides are a special case of the more general inhomogeneous waveguides, and the discussion applies to homogeneous waveguides with some obvious simplifications.

III. ELEMENTARY GROUP THEORY

By a group G is meant a set of distinct elements for which a combining operation is defined and which satisfies four group postulates [8, pp. 6–7]. The combining operation is called "group multiplication" and associates a third element of the set with any ordered pair of elements. The four group postulates are as follows.

1) The product of any two elements of G is itself a member of G.

2) The associative law holds so that for any three elements A, B, C of G; $(AB)C = A(BC) = ABC$.

3) G contains an element E, called the identity element, such that for any element A of G, $AE = EA = A$.

4) For any element A of G, there exists an element of G called the inverse of A, and denoted by A^{-1}, such that $A^{-1}A = AA^{-1} = E$.

The number of distinct elements of G is called the order of the group and denoted by g. For any particular group one can write a group multiplication table which displays the results of multiplying any two elements of the group. Note that group multiplication is not required to be commutative; that is, in general, $AB \neq BA$.

Examples of groups are provided by the sets of spatial symmetry operations discussed in the previous paper. It is easy to see that the set of n distinct rotations about an axis which was labeled C_n in [I] satisfies the group postulates. Likewise, the set of n distinct rotations about an axis and n mirror reflections in planes containing the axis which was labeled C_{nv} in [I] also satisfies the group postulates. Sets of spatial symmetry operations which satisfy the group postulates are called symmetry groups; for a discussion of uniform waveguides, only the C_n and C_{nv} symmetry groups need be considered.

The relationship of the group of spatial symmetry operations belonging to a particular symmetry group possessed by a particular waveguide and the modal electromagnetic fields of the waveguide can be expressed in either of two ways. Consider some symmetry operation R belonging to the symmetry group G. One can apply the symmetry operation R to the waveguide structure, leaving the modal fields fixed in space; or one can apply the symmetry operation R to the modal fields, leaving the waveguide structure fixed in space. In either case, after the symmetry opera-

[1] The results listed in paper [I] actually hold for more general waveguides. For example, they hold for inhomogeneous waveguides with uniaxial, piecewise-homogeneous media when the optical axis is parallel to the z axis, and also for waveguides with isotropic media where the media may be transversely inhomogeneous. For these more general cases, the analysis must be modified somewhat, but the results are the same as those cited in [I].

tion is applied, the modal fields must again be a solution to the boundary value problem for the waveguide. For clarity, we distinguish between symmetry operations on the structure and on the electromagnetic fields by defining $P(R)$ to be that symmetry operation acting on the electromagnetic fields which is equivalent to a spatial symmetry operation R on the structure. In order for the resulting electromagnetic-field–waveguide-structure relationship to be the same after operation by either R (on the structure) or $P(R)$ (on the electromagnetic fields), one must have

$$P(R)\boldsymbol{E}(r) = \boldsymbol{E}(R^{-1}r)$$

where $\boldsymbol{E}(r)$ is the electric field and R^{-1} is the symmetry-operation inverse to R; a similar relation holds for the magnetic field [8, p. 32].

In addition to symmetry groups there are many other sets of elements which satisfy the requirements for a group. Particularly important examples for symmetry analysis are sets of square matrices which satisfy all the group postulates with matrix multiplication as the group multiplication operation. Such a set of matrices is called a group representation, and certain group representations are central to symmetry analysis.

Given any symmetry group G of order g, one can always devise a set of g matrices which satisfies the same multiplication table as the symmetry group, after making a correspondence between each element of the symmetry group and one of the matrices. In fact, the number of possible group representations (sets of matrices) corresponding to any symmetry group is infinite. The simplest group representation for any symmetry group is a set of one-dimensional matrices of unit amplitude.

Although an infinite number of group representations can be written for any symmetry group, it is found that all of these can be written as the sum of a few group representations whose matrices have a dimension of one, two, or at most, three [8, pp. 19–20]. These few group representations are called the irreducible representations associated with the symmetry group. For the symmetry groups of current interest, the associated irreducible representations are known and tabulated (see, for example, the tables in [6], [7], or [8]).

The boundary value problems associated with waveguides can usually be formulated in terms of an eigenvalue problem. Typically,

$$L\psi = \lambda\psi$$

where L is an operator, λ is an eigenvalue, and ψ is the associated eigenfunction. Suppose the waveguide has the symmetry group G. If R is one of the symmetry operations of the group, then the operator $P(R)$ must commute with the operator L. Therefore,

$$P(R)L\psi = P(R)\lambda\psi$$
$$L(P(R)\psi) = \lambda P(R)\psi.$$

Thus if ψ is an eigenfunction with eigenvalue λ, then $P(R)\psi$ must also be an eigenfunction with eigenvalue λ.

If the eigenvalue λ has p degenerate eigenfunctions, ψ_i ($i = 1,2,\cdots,p$), then $P(R)\psi_n$, where ψ_n is one of these p eigenfunctions, can always be expressed as a sum over the p degenerate eigenfunctions. The effect of $P(R)$ is completely characterized by its effect on each of the basis functions ψ_i. For example

$$P(R)\psi_j = \psi_1\Gamma(R)_{1j} + \psi_2\Gamma(R)_{2j} + \cdots + \psi_p\Gamma(R)_{pj}. \quad (1)$$

The coefficients $\Gamma(R)_{ij}$ in these equations can be considered to be the elements of a $p \times p$ square matrix $\Gamma(R)$. If the ψ_i are collected into a row matrix

$$\tilde{\psi} = (\psi_1\psi_2\psi_3\cdots\psi_p)$$

then (1) can be written as

$$P(R)\tilde{\psi} = \tilde{\psi}\Gamma(R). \quad (2)$$

Any solution of the eigenvalue problem with eigenvalue λ must be expressible as a linear combination of the p independent solutions $\psi_1,\psi_2,\cdots,\psi_p$. Thus there is an equation analogous to (2) for every member of the symmetry group G. The complete set of matrices $\Gamma(R)$ for all g members of the symmetry group forms a representation. The basic assumption of symmetry analysis is the Irreducibility Postulate ([7, pp. 183–184] or [8, p. 34]):

> Provided there are no accidental[2] degeneracies, every degenerate group of eigenfunctions of an operator L provides an irreducible representation of the group of symmetry operations which leaves L invariant.

Thus the $\Gamma(R)$ in (2) form an irreducible representation. An alternative form of this postulate is the one which is used as the basis for the symmetry analysis here.

> For every p-dimensional irreducible representation of the symmetry group under which an operator L is invariant, we can find p-fold degenerate sets of eigenfunctions. Any further degeneracy would be accidental and expected to occur only rarely.

As a consequence, any eigenfunction of the operator L can be associated with a row of one of the irreducible representations of the symmetry group G. For those irreducible representations which are one-dimensional, each of the associated eigenfunctions is nondegenerate. For those irreducible representations which are two-dimensional, the associated eigenfunctions must occur in degenerate pairs, with one member of each pair associated with the first row and the second member with the second row of the irreducible representation. A similar statement applies to higher dimensional irreducible representations, but for uniform waveguides only one- or two-dimensional irreducible representations are encountered.

Suppose one finds a function ϕ which is a solution of the

[2] The fundamental assumption is adopted that the basic cause of mode degeneracy is (almost) always symmetry related. If a degeneracy is found which appears not to be symmetry related, it is termed an "accidental" degeneracy. In most cases, however, a deeper analysis reveals a subtle symmetry which produces the "accidental" degeneracy.

eigenvalue problem; ϕ may be a single eigenfunction or some sum of eigenfunctions. The function ϕ can be decomposed into a sum of functions, each of which belongs to one row of one of the irreducible representations of the symmetry group G by using the "projection operators" of the symmetry group [8, pp. 39–41]. When the projection operator $\rho_{kk}^{(i)}$ for the ith irreducible representation is applied to the function ϕ, it selects out that part of ϕ which belongs to the kth row of the ith irrducible representation.

For example, suppose

$$\phi = \sum_{j=1}^{N} \sum_{m=1}^{d_j} \psi_m^{(j)}$$

where the sum on j is over the N irreducible representations of the symmetry group, d_j is the dimension of the jth irreducible representation, and $\psi_m^{(j)}$ is an eigenfunction belonging to the mth row of the jth irreducible representation. Then

$$\rho_{kk}^{(i)} \phi = \psi_k^{(i)}.$$

For irreducible representations with $d_j \geq 2$, the eigenfunctions belonging to the several rows of the same irreducible representation will be degenerate with each other.

IV. APPLICATIONS TO UNIFORM WAVEGUIDES

In the brief discussion of Section III, it was stated that each eigenfunction of an operator can be associated with a row of one of the irreducible representations of the symmetry group to which the operator belongs. For uniform waveguides, the operator is $(\nabla_T^2 + k_i^2)$, and the symmetry group is either C_n or C_{nv}. The eigenfunctions are the E_z, H_z pairs for each mode of the waveguide. Thus each mode of a uniform waveguide can be identified with a row of one of the irreducible representations of the symmetry group of the waveguide. The mode classes of the uniform waveguide are defined on this basis. All of the modes belonging to the same row of the same irreducible representation are placed in the same mode class.

Thus the total number of mode classes for a uniform waveguide is equal to the total number of rows of all of the irreducible representations of the symmetry group of the waveguide. Further, every irreducible representation which has a dimension of two will have two mode classes associated with it whose modes are mutually degenerate. Since the symmetry groups C_n and C_{nv} have no irreducible representations with dimension higher than two, there will be no symmetry-induced modal degeneracies higher than two. This discussion is the basis for [I, tables 1 and 2].

In [I, sec. V] waveguides with C_4 and C_{6v} symmetries were discussed as examples (see [I, figs. 5(a) and 6(a)]. Reference to tables of irreducible representations of the symmetry groups (see [6], [7], or [8], for example) reveals that symmetry group C_4 has two one-dimensional and one two-dimensional irreducible representations, and symmetry group C_{6v} has four one-dimensional and two two-dimensional irreducible representations. Therefore, waveguides with C_4 symmetry have two nonedegenerate mode classes and a pair of mutually degenerate mode classes, and waveguides with C_{6v} symmetry have four nondegenerate mode classes and two pairs of mutually degenerate mode classes.

Using the projection operators introduced above, the azimuthal symmetry of the modes in any mode class can be determined. The azimuthal symmetry for each mode class is the characteristic that physically distinguishes the various mode classes. To exploit the projection operators of the symmetry group of the waveguide, one starts with a general representation for the longitudinal electric and magnetic fields in the waveguide and projects out that portion belonging to a particular row of a particular irreducible representation. The resulting expression is a representation of the modal field for the mode class associated with that row of that irreducible representation.

For waveguides with C_n symmetry, the exponential form of Fourier series is most convenient.

$$E_z(\theta,r) = \sum_{m=-\infty}^{\infty} A_m(r) \exp(jm\theta)$$

$$H_z(\theta,r) = \sum_{m=-\infty}^{\infty} B_m(r) \exp(jm\theta).$$

By applying the projection operators for each irreducible representation of a symmetry group C_n, the general form for the longitudinal electric and magnetic fields for the modes in each mode class can be obtained. This process was followed to determine the Fourier series representations of [I, table III] and the waveguide sectors shown in [I, fig. 5 and table V].

For waveguides with C_{nv} symmetry it is most convenient to write the Fourier series for the longitudinal electric and magnetic fields in the form

$$E_z(r,\theta) = \sum_{m=0}^{\infty} (A_m(r) \cos(m\theta) + C_m(r) \sin(m\theta))$$

$$H_z(r,\theta) = \sum_{m=0}^{\infty} (B_m(r) \cos(m\theta) + D_m(r) \sin(m\theta)).$$

By applying the projection operators for each irreducible representation of a symmetry group C_{nv}, the general form for the longitudinal electric and magnetic fields for the modes in each mode class can be obtained. This process was followed to determine the Fourier series representations of [I, table IV] and the waveguide sectors shown in [I, fig. 6 and table VI].

V. NONSPATIAL SYMMETRY

In [I, sec. VI], nonspatial symmetries were mentioned. The case of frequency-reversal symmetry will be briefly discussed here. This symmetry is based on the real-time-function postulate (Carlin and Giordano [10]) which states that the response of a system to an excitation which is a real function of real time must also be a real function of real time. Landau and Lifshitz [11] have shown that for exp $(j\omega t)$ time dependence this postulate requires that

$$\epsilon^*(-\omega) = \epsilon(\omega)$$

$$\mu^*(-\omega) = \mu(\omega)$$

for real ω.

The frequency-reversal operator $P(\Omega)$ is defined bv

$$P(\Omega)F(\omega) = F^*(-\omega).$$

Note that this is an antilinear operator, since

$$P(\Omega)[aF(\omega)] = a^*P(\Omega)[F(\omega)].$$

The full symmetry group of any uniform waveguide of the type considered in these papers includes, in addition to the spatial symmetry operations, the frequency-reversal operation plus the product of this operation with each of the spatial operations. Thus the total number of symmetry operations of the group is twice the number of purely spatial symmetry operations; and half of the total number of symmetry operations are antilinear. Because of the antilinear nature of these operations, it is not possible to find matrix representations of the complete symmetry group that satisfy the desired combining rules. It is possible, however, to find a set of matrix representations which satisfy a different set of combining rules; this set of matrices is called a corepresentation [8, p. 144].

A discussion of corepresentations is not feasible here, and only the results of interest will be mentioned. It can be shown that for most purposes [8, p. 145], only the usual irreducible representations associated with the subgroup of the complete symmetry group containing the spatial symmetry operations need be considered, with a few restrictions. For those symmetry groups of spatial operations whose irreducible representations are real (this includes all the C_{nv} groups), the inclusion of the frequency-reversal operation has no effect. For these cases the conclusions reached previously (ignoring the frequency-reversal operation) are all valid.

For those symmetry groups of spatial operations whose irreducible representations are complex, and where pairs of these irreducible representations are complex conjugates; then with regard to mode degeneracies, pairs of one-dimensional complex-conjugate irreducible representations act as two-dimensional irreducible representations. This applies to all of the C_n groups for $n > 2$, where irreducible representations with complex elements appear. Use of this artifice gives all of the results of interest to these papers without having to resort to the theory of corepresentations.

VI. CONCLUSIONS

The application of symmetry analysis to uniform waveguides enables one to: classify the modes of the waveguide into mode classes based on the azimuthal symmetry of the modal fields, predict the degeneracies of the various mode classes, describe the azimuthal symmetry of all the modes in a mode class, and determine the minimum waveguide sectors, and their associated boundary conditions, which are necessary and sufficient to completely determine the modes in a mode class. These results follow from a knowledge of the symmetry type of the waveguide under consideration, and they do not require a solution of a boundary-value problem.

The results obtained here are based on the theory of group representations, and in particular, on the set of irreducible representations associated with each symmetry group. Since a mode class can be associated with each row of each irreducible representation belonging to the symmetry group of the waveguide, the total number of mode classes is equal to the total number of rows of all of these irreducible representations. Further, the number of nondegenerate mode classes is equal to the number of irreducible representations with only a single row (that is, these representation are matrices of order one). The number of degenerate mode-class pairs is equal to the number of irreducible representations with two rows, Since no irreducible representations with more than two rows can occur for symmetry groups C_n and C_{nv}, there can be no symmetry-induced mode degeneracies of higher order than two.

The use of the projection operators obtained from the irreducible representations enables one to project out from a general function of the azimuthal coordinate the specific azimuthal variation characteristic of all of the modes in a particular mode class. From this, one can find the azimuthal symmetry possessed by the modal electromagnetic fields of all the modes in the particular mode class. This, in turn, leads to the determination of the minimum waveguide sector, and its associated boundary conditions, which is necessary and sufficient to completely determine all the modes in that mode class.

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